Constantly Rebalanced Portfolio – Is Mean-Reverting Necessary?

John M. Mulvey (Corresponding Contributor)

Contact Address:
Bendheim Center for Finance
26 Prospect Avenue (Dial Lodge)
Princeton University
Princeton, New Jersey 08540

Telephone: 609-258-5423
Fax: 609-258-0771
E-mail: mulvey@princeton.edu

Woo Chang Kim

Contact Address:
E-312, E-Quad, ORFE
Princeton University
Princeton, New Jersey 08540

Telephone: 609-258-6298
E-mail: wookim@princeton.edu

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John M. Mulvey and Woo Chang Kim

Princeton University

Princeton, NJ, USA

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Abstract

One of advantages of adopting multi-period portfolio models is improved investment performance via the fixed mix rule, which is called rebalancing gains or volatility pumping. Due to its similarity to the "buy low, sell high" strategy, it is often misunderstood that it requires mean-reverting processes for assets. In this paper, however, we show that mean-reversion is not necessary to benefit from the fixed mix rule, via a simple multi-dimensional geometric Brownian motion. We also list practical examples which are successfully implementing the rule in the domain of equities, commodity futures, alternative investments, and momentum strategies.

INTRODUCTION

There are a number of advantages of adopting multi-period models over traditional single-period, static models in portfolio managements [1]. One of more important benefits, among others, is improved performance on portfolio investments via the fixed mix rule [2-4]. The buy-and-hold rule, which represents single-period models, does not rebalance the portfolio at any intermediate juncture; hence the weight on each component might change as asset prices fluctuate in different proportions. In contrast, when a portfolio is constructed based on the fixed mix rule, it is rebalanced at every time point so that component weights remain the same as the initial state. In order to keep the weights unchanged, investors should sell assets whose prices have gone up, and buy ones whose prices have dropped. Therefore, in some sense, the fixed mix rule is analogous to the "buy low, sell high" strategy. Possibly due to such an analogy, there seems to be a widely spread misconception regarding the fixed mix strategy and its benefits – it requires mean-reverting processes for assets. Of course, because of its nature, it is not hard to see that it would be helpful to have such processes to achieve better performance. However, the truth is that mean-reversion is *not* necessary for the fixed mix to accomplish superior performance.

THEORETICAL BACKGROUNDS

We first recall performance of the buy-and-hold strategy. Suppose there are n stocks whose mean return is $r \in \mathbb{R}^n$ and covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$. Assuming normality, r^{BH}, the average buy-and-hold portfolio return with weight $w \in \mathbb{R}^n$, is normally distributed with mean $w^T r$ and variance $\sigma_p^2 = w^T \Sigma w$. That is,

$$r^{BH} \sim \mathrm{N}\left(w^T r, \sigma_p^2\right) \equiv \mathrm{N}(w^T r, w^T \Sigma w). \tag{1}$$

Next, let's consider a fixed mix portfolio constructed from the same stocks with the same weight

(*w*) as the previous buy-and-hold portfolio. Since it is rebalanced at every intermediate juncture,

it is required to model stock prices as processes. Thus, we model them as an *n*-dimensional

geometric Brownian motion whose return distribution for a unit time length would be the same

as the previous case. Then, the price process of stock *i* can be written as the following SDE.

$$\frac{dS_t^i}{S_t^i} = \left(r_i + \frac{\sigma_i^2}{2}\right)dt + dB_t^i, \tag{2}$$

where σ_i^2 is the *i*-th diagonal term of Σ (hence, variance of stock *i*) and for the Cholesky

factorization of Σ, L and the standard n-dimensional Wiener process $(W_t^1 \quad \cdots \quad W_t^n)^T$,

$$d(B_t^1 \quad \cdots \quad B_t^n)^T = Ld(W_t^1 \quad \cdots \quad W_t^n)^T. \tag{3}$$

Since the fixed mix portfolio is rebalanced at each time point to the initial weight (*w*), its

instantaneous growth rate is the same as the weighted sum of instantaneous growth rates of the

stocks at any given juncture. Therefore, the SDE for the portfolio wealth can be written as

$$\frac{dP_t^{FM}}{P_t^{FM}} = \sum_{i=1}^n w_i \frac{dS_t^i}{S_t^i} = \sum_{i=1}^n w_i \left\{\left(r_i + \frac{\sigma_i^2}{2}\right)dt + dB_t^i\right\}. \tag{4}$$

With simple algebra, one can show that, for the standard 1-dimensional Wiener process W_t,

$$\frac{dP_t^{FM}}{P_t^{FM}} = \left(w^T r + \frac{1}{2}\sum_{i=1}^n w_i \sigma_i^2\right)dt + \sigma_p^2 dW_t. \tag{5}$$

Hence, the return of the fixed mix portfolio for a unit time length can be given as

$$r^{FM} \sim N\left(w^T r + \frac{1}{2}\sum_{i=1}^n w_i \sigma_i^2 - \frac{1}{2}\sigma_p^2, \ \sigma_p^2\right) \equiv N\left(w^T r + \frac{1}{2}\sum_{i=1}^n w_i \sigma_i^2 - \frac{1}{2}w^T \Sigma w, w^T \Sigma w\right). \tag{6}$$

Therefore, returns of both buy-and-hold (r^{BH}) and fixed mix (r^{FM}) are normally distributed with

the same variance (σ_p^2), while the mean of the latter contains extra terms, $(\sum_{i=1}^n w_i \sigma_i^2 - \sigma_p^2)/2$.

These extra terms, which are often referred to as *rebalancing gains* or *volatility pumping*,

represent the value of having an option to constantly rebalance the portfolio to initial weights.

To observe its effects more closely, let's consider the following simple example: suppose we have n stocks where the expected return and the volatility of each are r and σ, and the correlation is given as ρ. Assuming the portfolio is equally weighted, the amount of the rebalancing gain,

$$RG = \frac{1}{2}\left\{\sum_{i=1}^{n}\frac{1}{n}\sigma^2 - \begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix} \Sigma \begin{pmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}^T\right\} = \frac{(n-1)\sigma^2(1-\rho)}{2n} \quad . \tag{7}$$

Now it is evident that the fixed mix strategy has benefit over the static buy-and-hold rule, *even without mean-reversion*; the rebalancing gain is always positive, except the case that all stock returns are perfectly correlated, in which it becomes 0. Note that the rebalancing gain is an increasing function of the number of stocks (n) and the volatility (σ), and is a decreasing function of the correlation (ρ). See Figure 1 for the illustrations of simulation results for the effects of σ, and ρ to rebalancing gains. Therefore, with the wisdom from the portfolio theory, one can see that volatile stocks should *not* be penalized when a portfolio is constructed with the fixed mix rule, as long as their correlations to other stocks are low and they possess reasonable expected returns; they can serve as good sources of rebalancing gains. The portfolio risks can be effectively reduced via dynamic diversification. For more complete discussion, see [2-6].

<Insert figure 1 about here>

PRACTICAL EXAMPLES

Under certain conditions, the fixed mix rule has been proved to be optimal in multi-period settings. Early on, Mossin [7] showed that it is the optimal strategy when an investor maximizes the expected power utility of her terminal wealth, assuming IID asset returns and no intermediate consumption. Samuleson [8] analyzed the problem in more generalized settings: Using an

additive inter-temporal power utility function of consumption over time, he proved that it is still

optimal to adopt the fixed mix rule when the investor is allowed to consume at intermediate

junctures. Merton [9] also concluded the same in the continuous time setting model.

Indeed, there are many practical applications which are successfully taking advantage of

rebalancing gains via fixed mix rules. Among others [10-12], one of good examples is the S&P

500 equal-weighted index (S&P EWI) by Rydex Investments (see Figure 2) [13]. Unlike

traditional cap-weighted S&P 500 index, it applies the fixed mix rule to the same stocks as S&P

500, rebalancing them every six month to maintain the equally weighted portfolio. During 1994

to 2005, S&P EWI earned 2% excess return with mere 0.6% extra volatility over S&P 500. This

added profit is partially due to superior performance of small/mid-sized stocks but also can be

accounted for by rebalancing gains.

<center><Insert figure 2 about here></center>

Implementations of the fixed mix rule could also lead to successful leverage. Figure 4 illustrates

levered portfolios of buy-and-hold and fixed mix portfolios constructed in two different domains.

In the figure, (A) compares efficient frontiers of buy-and-hold and fixed mix portfolios, which

are constructed with six traditional assets (S&P 500, EAFE, Lehman long-term bond index,

Strips, NAREIT, and Goldman Sachs commodity index) and four alternative assets (hedge fund

index, managed futures index, Tremont long-short equity index and currency index) [14]. Both

are equally weighted and levered up to 100% via t-bill rate. While the buy-and-hold portfolio is

not rebalanced, monthly rebalancing rule is adopted for the fixed mix for the entire sample

period (1994~2005). Also, (B) in Figure 4 depicts results from portfolios of industry-level

momentum strategies across international stock markets for 27-year sample period (1980~2006)

[15]. Momentum strategies are constructed in five non-overlapping regions (U.S., E.U., Europe except E.U., Japan, and Asia except Japan), and aggregated into equally weighted portfolios with leverage up to 100%. Similar to the previous case, the fixed mix portfolio is rebalanced monthly. In both cases, the efficient frontiers from the fixed mix dominate ones from the buy-and-hold.

<Insert figure 3 about here>

IMPLEMENTATION ISSUES

The fixed mix rule is now becoming a norm in various financial domains. For instance, it is now commonplace for large pension plans, such as TIAA-CREF, to automatically rebalance client-selected portfolios back to client-selected weights, at the client's requests. Given the circumstances, it is imperative to address issues regarding practical implementations. First, since the best sources of rebalancing gains are volatile financial instruments with low intra-correlations, it is crucial to find a set of relatively independent assets. However, the task is very unlikely to be perfectly achieved in the real world. Second, even such a set exists at certain time point, correlations could change over time. For instance, it is well-known that stock indices across international markets become highly correlated upon serious market distress. Also, one should consider transaction costs such as capital gain taxes upon deciding the rebalancing intervals. Although frequent rebalancing could lead to investment performance close to the theoretical values, it may deteriorate performance due to transaction costs. Careful analysis on this trade-off is required. Good references regarding practical implementations of the fixed mix rules include [16-19].

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FIGURES AND CAPTIONS

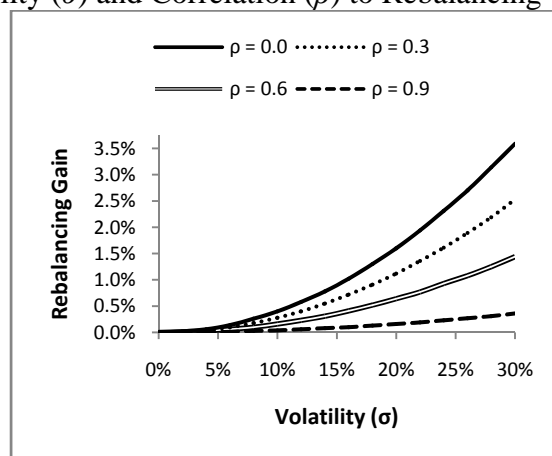Figure 1: Effects of Volatility (σ) and Correlation (ρ) to Rebalancing Gains ($n = 5$)

Figure 2: Log Prices of S&P 500 and S&P EWI during July 2003 to December 2006
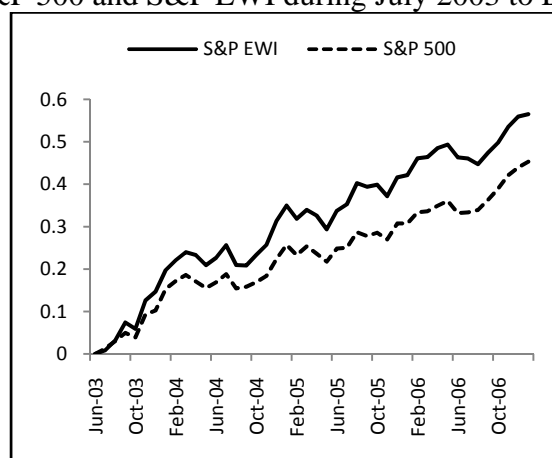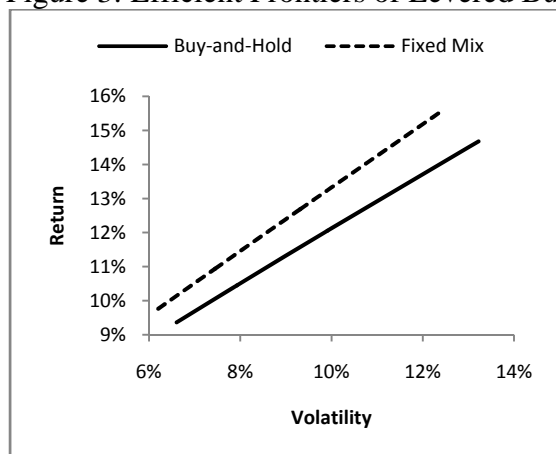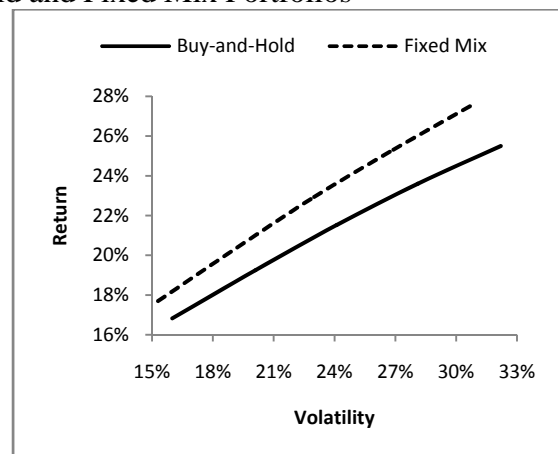

Figure 3: Efficient Frontiers of Levered Buy-and-Hold and Fixed Mix Portfolios

(A) mix of traditional and alternative assets (1994-2005)

(B) mix of momentum strategies of 5 regions (1980-2006)